Teaching Differential Equations: concepts and applications

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Abstract

A multidisciplinary group of investigators and professors of different fields, from technologies of education to engineering, has been linked in order to understand the sequential way how the differential equations are taught. Starting with the introduction of the derivative concept, in the secondary level, till the numerical solution of differential equations, at the university, without forgetting the analytical solution of the equations, it allows a reflection on the learning processes. This reflection will permit to establish the bridge between the secondary and the university. In the present work, it was intended to identify the difficulties felt in the teaching/learning process in this particular area, identifying the informatics applications that are used and how, in a way that can improve students' motivation.

I. INTRODUCTION

It is generally accepted that computer based problem solving is a very important application of computers in engineering education [1][2], but there is still some practical difficulties in its implementation and use. The use of computational methods for solving differential equations is crucial and it is one of the core subjects of the post graduation courses of the Engineering School at the University of Minho.

Real life engineering problems can be usually described by sets of differential equations that are mathematical approximations of the physical reality. Normally these equations include one or more differential equations and a set of initial/boundary conditions. The set of differential equations can be solved by complex analytic methods if some simplifications are introduced, but they often are solved using numerical algorithms implemented in computer programs [3]. The concept of a differential equation is introduced in high school, and it will be a major topic of study throughout university especially in the science and engineering degrees.

The introduction of the concept of these differential equations as a means to model and simulate the real world is not always an easy task. To be effective in doing so, there has been an effort to introduce associated methodologies in the base formation of engineering students as soon as possible.

Students in the first years of engineering degrees show some of deficiencies both in the modeling of physical systems as well as in the solving of the differential equations that are an important part of that model. Current research in education, namely constructivist based theories [4], has demonstrated that effectiveness of learning is largely dependent on the ability to promote the immersion of the learner in authentic situations. According to this view, learning in general and learning to solve problems is a process of individual and collaborative exploration addressed to the real contexts of implementation.

Immersion in real situations, contextual learning, and collaborative learning are the main dimensions of this project of learning object development in the field of differential equations.

Although the divulgation and the successive introduction of computers in teaching, Mathematics is still a subject where students obtain poor results, which can be seen in the national exams. This shows that somewhere between high school and the university students no longer are able to acknowledge the necessary basis to be successful in a technology degree such as engineering.

It is in this context that the actual project was conceived: combining the synergies of different learning areas, namely with teachers of different levels of teaching, pedagogic experts and researchers of teaching with new technologies, we intended first of all to understand how the various concepts are gradually taught, from the derivative of a function to the modeling of complex physical systems involving ordinary differential equations (ODEs) and/or partial differential equations (PDEs). In this sense, the contents of the different levels of mathematics were collected trying to identify its main difficulties [5].

Some effort was done in order to diminish the difficulties students have in the engineering degrees when trying to apply their knowledge in project implementation.

It is intended to understand how the concepts are sequentially taught since the introduction of the derivative of a function in high school, until numeric solution of problems involving differential equations, passing through the analytic solution of these problems. Having all these into account, it is our objective to develop new learning/teaching environment contents to be explored in the domain of Mathematics.

The creation of new informatics applications with high level of interaction with the user constitutes the ideal work environment for the student, in the process of construction of the representation and elaborations of new concepts, covering some of the flaws of the existent commercial software.

As a starting point, the multidisciplinary group, have been looking for a means to integrate technologies of information in the curricular processes of teaching/learning for the concept of derivative and its applications in many areas of engineering [6]. These learning objects [7], learning environments in a computer are flexible systems that allow a better link with real problems and the progressive knowledge construction in the way to solve of new problems. However, the use of new technologies when teaching differential equations constitutes a hard task in redefining the contents, defining the demands of simplicity necessary to use the software, as well as the development of graphical interfaces [8-11].

In this paper, we will present and discuss some of the existent software tools used by the members of our teams when teaching maths, or related subjects.

In the secondary school, the introduction to new technologies has been a slow process, not only because there is the need to build new practices and teaching sets, but fundamentally because it is hard and expensive to allow a general use of these new didactic tools.

In the university, the use of commercial software such as MatLab® or MAPLE® is a lot more common. Notice that there are simplified freeware versions of these tools. Both MatLab® and MAPLE® allow us to solve differential equations using pre defined functions with that purpose. You can obtain a very friendly environment where the results can easily be seen and interpreted in a graphic form. The computer learning environments is a flexible system that allows a better link with real world problems and the progressive learning construction when solving new problems [12]. Its use can become easier in new graphical interfaces are developed [8-9], [11]. However, the development of graphical interfaces is an involving and complex task that requires tests of usability during its development. The complete knowledge of the pros and cons of the existent software tools will allow us to recognize the need to design and develop tools easier to use and applicable to all the levels of teaching.

This way, our focus will not be on the amount of knowledge to be learned, but on the way to use the knowledge.

II. CONTENTS CATEGORIZATION AND REFLECTION

During this project development, was taken a work of categorization of information concerning the derivative concept and its use is taught in several levels of education. In secondary education, the concept of the derivative of a function is taught by given typical examples of applications in the fields of physics. In the university, the students learn the analytical solutions and the numerical solution of differential equations, using softwares.

This work allowed identifying the contents to be enclosed in software and still intended to create a bridge between secondary and the university.

A. Derivative concept

When differential calculus is taught at the high school level, the main objective is to develop the derivative concept through different representations: numerical, graphical and symbolic. The derivative concept is a typical example of a sequential teaching through the different levels of high school. It starts at the 11th level with “introduction to differential calculus I” and it is followed by “Introduction to differential calculus II” at the 12th level.

The main topics related with the derivative concept are: limit, function, variation rate, linear equation and tangent line. The sequential teaching of the derivative concept is related to the fact that students need this concept in two different moments: to calculate the tangent to a curve and to calculate the area below a curve.

The main learning difficulties have been identifyed and they can be summarized as follow:

1. understand the derivative meaning when the derivative is represented by two parts
2. apply the derivation rules
3. choosing the parameters when working with partial differential equation

4. solve maxima and minimum problems

It can also be observed that different concepts from Physics like, conservation laws, particle movement, use the differential calculus.

B. Analytic and numerical solution of differential equations

In the first two years of the engineering degrees, there is a set of subjects in the areas of mathematics and physics, where the derivative of a function concept and its many applications are studied in detail. This typically happens in Calculus.

The study of functions of one or more variables when, for instance, local maximums/minimums location, the study of Taylor or MacLaurin series of a function in the vicinity of a given point are some of the many examples where the concept of derivative is necessary and important. Also, in many engineering problems, it is necessary to solve either analytically or numerically ODEs or PDEs, and once again the concept of derivative is present. To solve problems that involve ODEs or PDEs, a series of methods to solve these equations analytically are presented and discussed with the students. The correspondent programmes are described in details elsewhere [5]. The main tools are still a piece of paper and a pencil although as mentioned before, it is more and more common to introduce softwares, namely MatLab® and Maple® (in more detail in the next section), to facilitate the students in their learning process.

When the analytical solution of the differential equations is not possible, the students will learn how to find alternative approximate solutions in a new subject: Numerical Methods (2^{nd} or 3^{rd} year depend of the Engineering course field). They also, will be able to estimate the error of that approximate solution. Several engineering problems are described by nonlinear differential equations and Numerical Methods are used to obtain the final solution.

III. LEARNING TOOLS

A. Secondary level: concepts and modeling

The access to the new technologies at high school level and the use of algebraic software, like Mathematica and ClassPad, is still very difficult due to financial problems. Also, the inclusion of these new technologies would modify and require new practices. This is a very slow process because new teaching equilibrium has to be created.

The use of graphical calculators is now starting to modify the contents and the teaching methods.

B. University: Analytic and numeric solutions:

In order to show some of the software available to solve differential equations, we will start by presenting a problem that can be seen in any physics book. The problem is called “free vibration without damping” [1]. It consists on having a mass-spring system where damping is negligible, and no external force is being applied, described by the differential equation:

$$m\ddot{x} + kx = 0 \tag{1}$$

where m (m>0) is the mass of the object, k (k>0) is the stiffness of the spring and x is the distance the spring is stretched. In order to finish the formulation of this problem, we will have to impose the initial conditions. Considering that the spring is stretched L, and the object is freed without any velocity, the two initial conditions are defined:

$$\begin{aligned} x(0) &= L \\ \dot{x}(0) &= 0 \end{aligned} \tag{2}$$

Since this ODE is a homogeneous second order differential equation with constant coefficients, with a well known analytical solution of the form:

$$x(t) = e^{\alpha t} \tag{3}$$

where α is a set of constants, either real or complex. By substituting it in the ODE that describes the problem, the characteristic equation associated with our ODE is obtained:

$$m\alpha^2 + k = 0 \tag{4}$$

According to the solutions of the characteristic equation, the solution of the ODE will be different. In this case, we have two different complex solutions for α, meaning that the solution for the ODE is:

$$x(t) = c_1 \cos(\sqrt{\frac{k}{m}}t) + c_2 sen(\sqrt{\frac{k}{m}}t) \tag{5}$$

Applying the initial conditions (2) that complete the formulation of the problem (1), the system of linear equations (6) is obtained.

$$\begin{cases} x(0) = c_1 \cos(0) + c_2 \sin(0) = L \\ \dot{x}(0) = -\sqrt{\frac{k}{m}} c_1 sen(0) + \sqrt{\frac{k}{m}} c_2 \cos(0) = 0 \end{cases} \tag{6}$$

The values $c_1 = L;\ c_2 = 0$ are the solution of this system of equations meaning that the solution to the problem (1-2) can be written as,

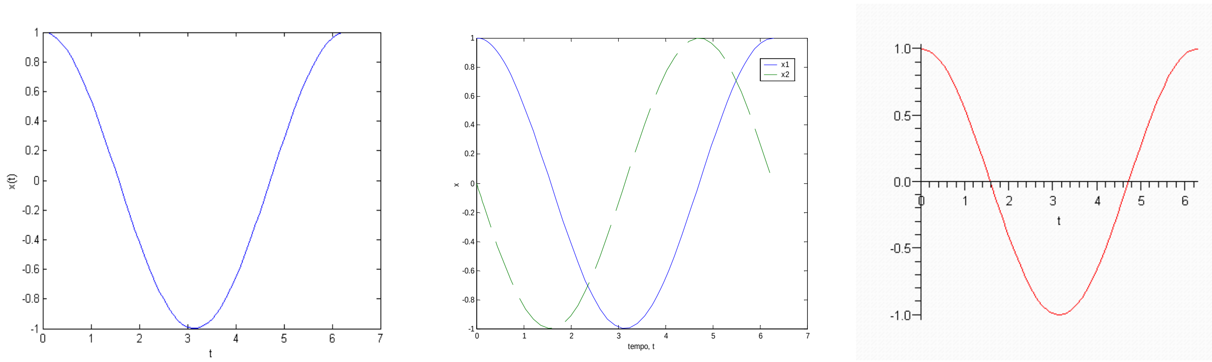
$$x(t) = L\cos(\sqrt{\frac{k}{m}}t) \tag{7}$$

MatLab®, Maple®, MAXIMA (a free version of MatLab®) and CoNum would be used in order to resolve the “free vibration without damping” problem (1-2) numerically.

MatLab® offers a wide variety of functions either analytical or numeric to solve ODEs and PDEs. The problem (1-2) can be easily solved using the “dsolve” function:

```
sol=dsolve(‘m*D2x+k*x=0’,’x(0)=L’,’Dx(0)=0’,’t’)
L=1; m=1; k=1; t=0:0.1:2*pi;
x=L*cos(t.*((1/m)^(1/2)*(k)^(1/2)));
plot(t,x); xlabel(‘t’); ylabel(‘x(t)’);
```

MatLab® also offers the possibility to, easily, represent graphically the results with the “plot” function. Initially the constants *L*, *m* and *k* are declares and the initial and final value of time *t*. The set of command gives the graphic representation of the spring stretch vs time (Figure 1.a).



(a) (b) (c)

Figure 1. Problem (1-2) solution obtained by using MatLab®: (a) analytic and (b) numeric, and (c) by using MAPLE®.

To numerically solve the physical problem (1-2) defined by a differential equation of 2nd order firstly it must be transformed into a system of two differential equations of 1st order, after a variable transformation. The pupils are taught to use the Matlab® m-files where the differential equations system is defined, and be solved by the use of a simple command “ode45”:

```
[t,x]=ode45(@mola_corpo,[0 2*pi],[1;0])
plot(t,x(:,1),t,x(:,2),'--');legend('x1','x2');
xlabel('tempo, t'); ylabel('x');
```

The final results can easily be observed in a graphical form (Figure 1.b), by using the “plot” command.

This problem can also be solved by using Maple®:

```
ode := m $diff(x(t), t, t)
   C k $x(t) = 0 ;

ics :=x(0) = L,
    D(x)(0) = 0;

dsolve( {ode, ics} );
```

```
plot( cos( t ) , t = 0 ..2*p );
```

The graphic representation obtained is illustrated in Figure 1.c.

Using the open source computer algebra system Maxima, the problem is now solved by the following set of commands:

```
(%i1) ode2(m*'diff(x,t,2)+k*x=0,x,t);
(%i2) ic2(%,t=0,x=L,diff(x,t)=0);
```

The advantage of the last software tool is that it is free and its download can be done through many sites in the internet, namely from its official site in http://maxima.sourceforge.net/.

The numerical solution of differential equations, the students can use a academic software, CoNum, with free access (http://www.norg.uminho.pt/emgpf/software/conum.zip). The book (in Portuguese) associated to the CoNum, contain all the algorithms of the numerical programs [13]. The CoNum was developed in the University of the Minho and is successfully used in all the Engineering courses. Through a main menu, the students can access to several submenus allowing, in accordance with the area of the problem in study, to choose the most adequate method (Figure 2). The input values are introduced in specific menus and the final results presented in a text window. The only inconvenience is the graphical representation. As it does not exist, the students normally use others software namely the EXCEL, for the graphical visualization of the obtained results.

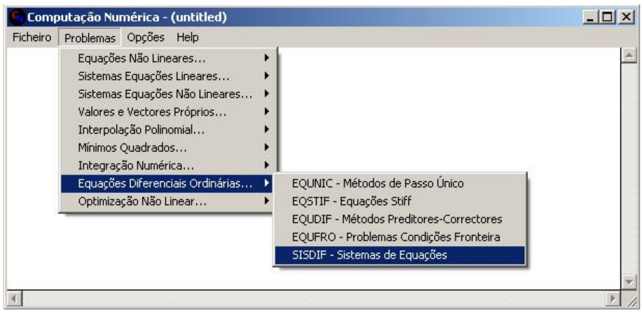


Figure 2. CoNum main menu (in portuguese).

This program can be used for the numerical resolution of the problem "free vibration without damping" (1-2). To solve the differential equation of 2nd order through CoNum, the student must firstly identify the problem type, ODE, and them the method to be used to solve the system. Then all the initial values that identify the problem are introduced: dimension, function vector, initial values, superior interval limit time, time step, and method type (Runge-Kutta or Preditor-Corrector). Then, the final results are presented in a window text.

C. *Tools for assessing students:*

The Department of Mathematics for the Science and Technology, at the University of Minho (Portugal) uses the Maple TA system for several mathematics courses taken by engineering students of the first year of college, for instance in Calculus and Linear Algebra [14].

We have large number of students attending these classes. Knowing that they are not obliged to be present, this large number of students creates conditions for implementing distance learning methods. Additionally, the Bologna education transformation decreases the classes' number of contact hours. To overcome this situation, new learning facilities were created contributing to help the students on their individual work.

Maple TA is not a tool designed specifically to deal with differential equations, still when it is well used, it can be a valuable student assistant since it allows to maximize the student's study on any field of science, for instance on differential equations. Note that this assessment tool is very powerful allowing generating many different questions of different types via the use of random variables. The defined problem (1-2) can be written in a way to be used in Maple TA, and, therefore, being available to assess the knowledge of the student in any part of the world [5].

IV. CONCLUSIONS AND FUTURE WORK

The contents and the methods used in the derivative and differential equation subjects teaching/learning categorization was the first step of a multidisciplinary group of investigators and professors of different fields, in order to help the students to overcome the difficulties that can appear when studying these subjects.

This work also allows the cooperation between teachers of different levels of teaching, pedagogical experts as well as researchers in TIC, in a way that it increase the discussion and sharing of ideas with the objective of improving the learning/teaching process.

In order to develop a more constructive approach when teaching differential equations, means creating a new teaching paradigm that emphasises the modeling of learning, the analysis of the differential equation in analysis, comprehension of the qualitative behaviour of the solution, the use of software tools with graphics facilities, and finally being able to "communicate mathematics" instead of "executing tasks or methods".

In the first stage, the students should know and understand the different techniques used to obtain the solutions for the different types of differential equations and being able to analyse them in order to present qualitative aspects of the solution.

In the second stage, a context should be given to the student in such a way that it facilitates the development of higher level of mathematical reasoning through modeling of physical phenomena and analysis of these models. Questions should be raised about the model, and theorems that can answer these questions should be analysed.

The use of software tools demands that a minimum level of aptitude in using these software tools, so that the students are able to focus on the concepts to explore and the questions to put, Implementing a laboratory of mathematics involves an increase in the amount of work to the teacher (writing manuals, strategies of control and guaranteeing the equipments work throughout the class, writing up good examples that can be explored, …). However, the students find that the classes are more interesting, and their level of confidence in terms of the solving of differential equations, and writing up a report are increased.

It seems obvious that this new paradigm includes a process of transmission/negotiation inserted in the theoretical classes dedicated to the translation between oral language, graphical representations and symbolic.

There are many software tools in the market that can help the students in their study of solving differential equations. Some of them are commercial such as MatLab® or MAPLE®, that not only help the students in their process of solving problems that involve differential equations but also have to possibility of representing graphically these solutions. However they are expensive.

There are also available some freeware tools such as Maxima, that being also very easy to use, don't have the same graphical potential.

Since one of the aims of this study is to search and analyse software tools to teach Mathematics, we have to mention that the Department of Mathematics for Science and Technology is implementing in a number of subjects the use of Maple TA. It allows the assessment of a student on line in any part of the world, giving immediate feedback to the student and the teacher. Its drawback once again is that it is an expensive tool.

We should emphasize that the use of new Technologies in teaching either differential equations or any other topic always constitutes a challenge in redefinition of contents and demands of simplicity in the software used.

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