On the Use of Mathematica in Engineering Education

Audun WEIERHOLT
Faculty of Engineering, Oslo University College, P.O. Box 4, St. Olavs plass, N-0130 Oslo, Norway.

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ABSTRACT: *The purpose of this report is to demonstrate how Mathematica can be used by students and educators to solve differential equations and simulate dynamic systems. The approach is purely educational and shows how one can gain insight in the salient features of a system by playing with system parameters and initial conditions. Many educators are concerned about the use of mathematical software: Students should learn to solve differential equations by hand. We agree on that. However, anyone who has worked to find particular solutions to inhomogeneous differential equations knows that this is a fairly laborious task, if the input is more complicated than a unit step function. The main advantage of a system like Mathematica is that students can do simulations and explore "what if?" situations by changing parameter values, and they can do calculations they would normally never do by hand. They get virtually instant feedback, and results can easily be plotted for illustration purposes. For the purpose of demonstration the well-known system of a mass attached to a spring and a damper has been chosen. It can be used for simulation of many vibrational systems. For example shock absorbers in a car. The differential equation governing this system is stated, and a few examples of solutions of homogeneous and inhomogeneous systems are presented. Attention is focused on the underdamped and critically damped systems, particularly on the build-up and dissipation of energy. Some of the calculations are not found in most textbooks.*

1 MASS AND SPRING AND DAMPER

The differential equation governing the motion of a mass attached to a spring and a damper is given by

$$m\frac{d^2 x[t]}{dt^2}+b\frac{dx[t]}{dt}+k\,x[t]=f[t] \qquad (1)$$

m is the mass, b is the damper constant and k is the spring constant. x[t] is the displacement from an equilibrium position, its first derivative with respect to time is the velocity of the mass and its second derivative is the acceleration. f[t] is an externally applied force. The homogeneous part of the solution is easy to find. Its form depends on whether the characteristic equation has two real roots, one real root or two complex conjugate roots. This is determined by the quantity $\sqrt{b^2-4km}$ which can be real, zero or imaginary. This corresponds to an overdamped system, a critically damped system and an underdamped system, respectively. To find the inhomogeneous solution one needs to know the form of the input f[t]. If this is a unit step function it is no problem, if it is a sine function it is still no problem, but the procedure is quite time consuming. Most students will do this only once as part of a homework assignment. If the input is more complicated than a sine function, but still periodic one may use Laplace transform methods. In most other cases one has to resort to numerical techniques. The point is: If one wants to solve some problem repeatedly by changing input, initial conditions or system parameters, a system like *Mathematica* is invaluable.

1.1 $b=\frac{1}{4}\sqrt{km}$ - Underdamped System

We first consider the underdamped case with no external force. The mass i displaced from equlibrium and released at t = 0. With $b=\frac{1}{4}\sqrt{km}$ the characteristic equation clearly has two complex conjugate solutions, and the solution of the differential equation will be of the form

$$x[t] = e^{-\alpha t}(A\cos(\beta t) + B\sin(\beta t)) \quad (2)$$

with β given by

$$\beta^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$$
$$\alpha = \frac{b}{2m} \quad (3)$$
$$\omega_0^2 = \frac{k}{m}$$

ω_0 is the natural angular frequency of oscillation of the undamped system. β is the natural angular frequency of oscillation of the damped system if the system is excited and then left to itself. It is worth noting that $\beta < \omega_0$. A and B are determined by the initial conditions.

```
Clear[x]
k = 100;
m = 1.;
x0 = 0.1;
b = 1/4 Sqrt[k m];
e1 = m x''[t] + b x'[t] + k x[t] == 0;
e2 = x[0] == x0;
e3 = x'[0] == 0;
s1 = DSolve[{e1, e2, e3}, x[t], t];
```

$\{\{x[t] \to 0.1\, e^{-1.25t}(1.\,Cos[9.92157\,t] + 0.125988\,Sin[9.92157\,t])\}\}$

```
x[t_] = x[t] /. s1;
```

The output from **DSolve** is of the form of equation (2).

```
Plot[Evaluate[x[t]], {t, 0, 2.5}, PlotPoints → 35, GridLines → Automatic, PlotRange → All,
 PlotStyle -> {{RGBColor[1, 0, 0], Thickness[0.012]}}]
```

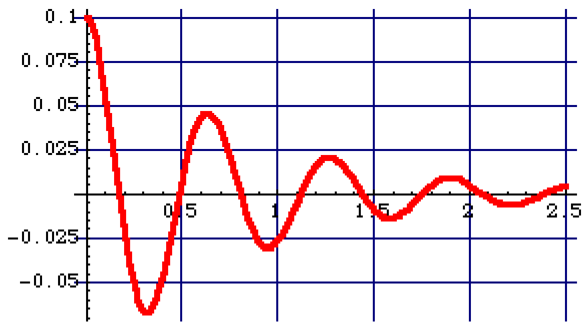

Figure 1: Natural response of underdamped system.

Displacement of mass as a function of time. The reason for this oscillatory behavior is that the system has two ways of storing energy. As potential energy when the spring is stretched or compressed and as kinetic energy when the mass has a velocity. The oscillations fade after a while and the system comes to rest. This is due to energy dissipation by the frictional element.

1.2 Energy considerations

The next figure shows the potential energy (solid line) and kinetic energy (dashed line). The amount of energy available at t = 0 is given by the displacement of the mass from its equilibrium position and is stored in the spring.

$$E_s = \frac{1}{2} k\, x[0]^2 \qquad (4)$$

```
Es = 1 / 2 k x0^2
```

0.5

When the mass is released the energy swishes back and forth between potental and kinetic energy with decreasing amplitude until it stops.

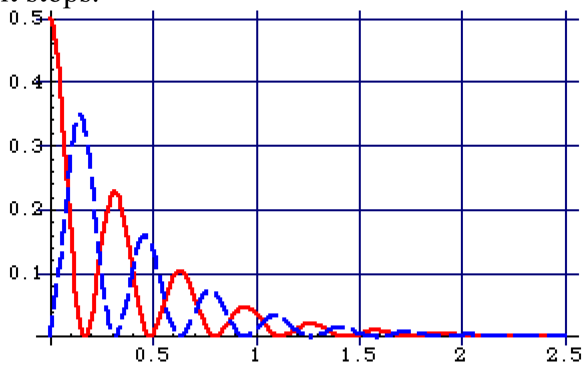

Figure 2: Solid line: Potential energy as a function of time. Dashed line: Kinetic energy as a function of time.

The amount of energy absorbed by the damper is given by

$$E_A = \int_0^{\infty} b \left(\frac{dx[t]}{dt}\right)^2 dt \qquad (5)$$

One can easily check if this is equivalent to the amount given in eq. (4). Since we have used **DSolve** we have an analytical expression for the solution which can be integrated to **Infinity** or ∞. So the integral of equation (5) can easily be evaluated.

```
Integrate[Evaluate[b x'[t]^2], {t, 0, ∞}] // Chop
```

{0.5}

This result compares favorably with the energy available in the spring at t = 0, see eq. (4). Thus, the students can see for themselves that the energy dissipated is identical to the energy stored. In principle it is not difficult to integrate equation (5) by hand, you just differentiate equation (2) with respect to time, square it and integrate term by term with respect to time. But it takes a lot of time, and most people will do this only once.

1.3 $b = 2\sqrt{km}$ - Critically Damped System

In this case the square root $\sqrt{b^2 - 4km}$ is zero. The characteristic equation has two identical real solutions and the solution of the differential equation will be of the form

$$x[t] = A e^{-\omega_0 t} + B t e^{-\omega_0 t} \qquad (6)$$

where ω_0 is the angular resonance frequency of equation (4).

```
Clear[x]
b = 2 Sqrt[k m];
e1 = m x''[t] + b x'[t] + k x[t] == 0;
e2 = x[0] == x0;
e3 = x'[0] == 0;

s2 = DSolve[{e1, e2, e3}, x, t] // Chop
```

$\{\{x \to \text{Function}[\{t\}, 1.\ e^{-10.\ t}\ (0.1 + 1.\ t)]\}\}$

```
x[t_] = x[t] /. s2;
```

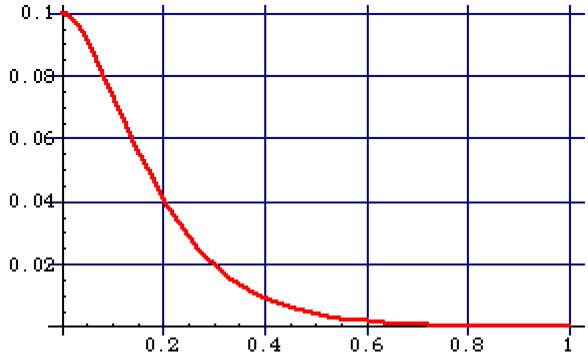

Figure 3: Displacement of mass as a function of time.

Natural response of critically damped system. Note the difference in scale on the horisontal axis compared to the previous example. In this case the system comes to rest without crossing the zero line. The system is said to be critically damped. If b is any smaller than in this example there will be an overshoot. Students can play around with initial conditions and see if they can make the solution cross the zero line. This should also be done by hand.

1.4 Energy considerations

It is interesting to study the potential and kinetic energies as functions of time. There is surprisingly little buildup of kinetic energy, just enough to move the mass nice and easy to its equlibrium position.

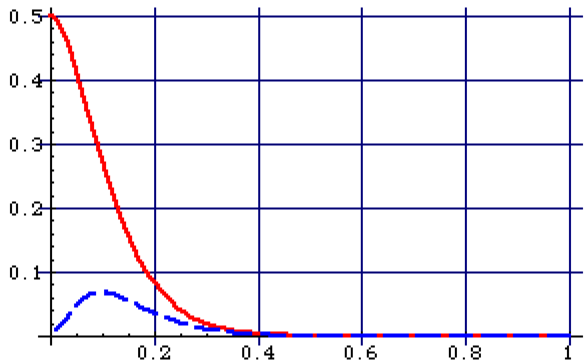

Figure 4: Evolution of potential (solid line) and kinetic (dashed line) energy as a function of time in critically damped system.

The amount of energy dissipated by the damper is given by

```
Integrate[Evaluate[b (x'[t])^2], {t, 0, Infinity}]
```

{0.5}

This is identical to the energy that was stored at the beginning.

1.5 $b = 3\sqrt{km}$ - Overdamped System

Similar calculations can be made when $\sqrt{b^2 - 4km}$ is real In this case the characteristic equation has two distinct real solutions and the solution of the differential equation will be of the form The results for an overdamped system will be qualitatively very similar to those of the critically damped system, and will not be considered further.

2 FORCED RESPONSE

The next step is to study how the mass, spring and damper system responds to an externally applied force f[t] see equation (1). The external force is taken to be

$$f[t] = A\,\mathrm{Sin}[\omega t] \tag{7}$$

We use the same parameter values as for the homogeneous equation

2.1 $b = \frac{1}{4}\sqrt{km}$ Underdamped System

There is no initial displacement nor initial velocity. The frequency is assumed to be ω_0 as given by eq. (3). Here a numerical solution is provided.

```
Clear[x]
ω = Sqrt[k/m];
A = 10;
b = 1/4 Sqrt[k m];
e1 = m x''[t] + b x'[t] + k x[t] == A Sin[ω t];
e2 = x[0] == 0;
e3 = x'[0] == 0;

s3 = NDSolve[{e1, e2, e3}, x, {t, 0, 5}]

{{x → InterpolatingFunction[{{0., 5.}}, <>]}}
```

The dashed curve in figure 5 below shows the displacement of the mass from its equlibrium position. This is the response of the system to the harmonic force. It takes about 4 seconds for this motion to reach a stationary state. The solid curve shows the displacement if the force was applied to the spring only, this is simply the force divided by the spring constant. In the stationary state the displacement is about four times larger than the spring - force relationship would suggest. One may also note that there is a time delay, or a phase shift, between the application of force and the response. The force reaches its maxima before the displacement does.

Now, the question is: why is the displacement so large? Again the fundamental reason is that the system has two ways of storing energy: potential energy in the spring and kinetic energy in the moving mass. And the frictional force is sufficiently small that it does not prevent a build-up. The situation is completely analogous to setting in motion a kid on a swing.

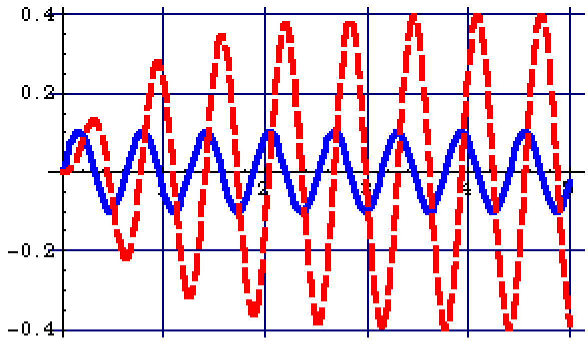

Figure 5: Dashed curve: Displacement of mass. Solid curve: Applied force divided by spring constant.

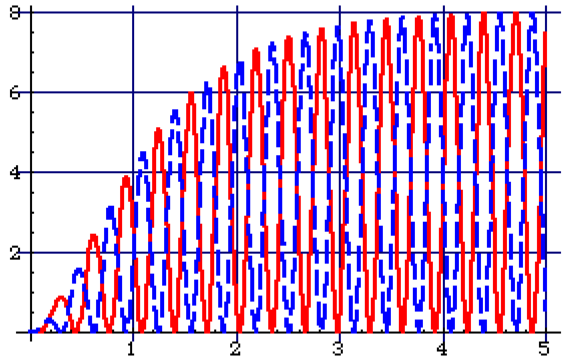

Figure 6: Potential energy (solid line) and kinetic energy (dashed line) as functions of time.

Figure 6 shows the build-up of potential energy (solid line) and kinetic energy (dashed line) as functions of time. The two forms alternately take the lead as they build up to a stationary value of about 8 Joules after 4 to 5 seconds. To see how this comes about we calculate the power supplied to the system by the external force as a function of time. This power is given by the following expression.

$$P_s[t] = f[t] \cdot \frac{dx[t]}{dt} \tag{8}$$

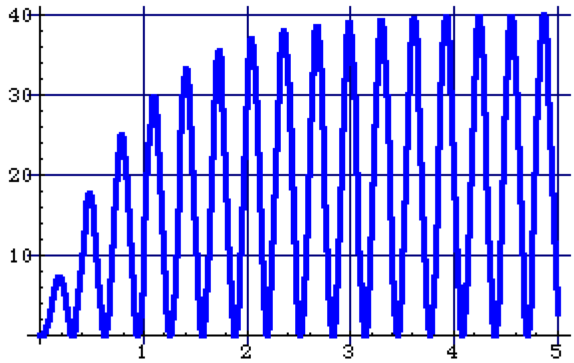

Figure 7: Power supplied to system by the external force as a function of time.

We also calculate the power dissipated in the damper as a function of time and plot the result. As expected, the power dissipation has its maxima whenever the kinetic energy does. See figure 6. The power dissipated by the damper is given by equation (5). The result is displayed in figure 8.

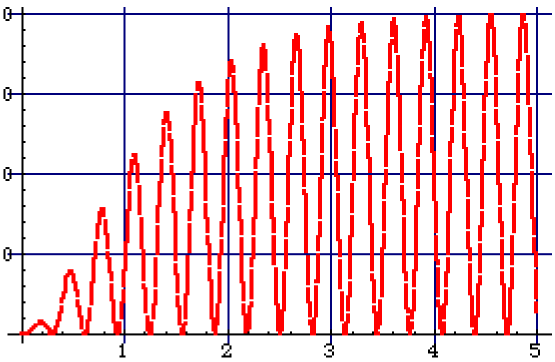

Figure 8: Power dissipated in damper as a function of time.

There seems to be a surplus of power being supplied over power being dissipated. The difference between the two is displayed in figure 9. This difference goes to zero as steady state approaches. In steady state the power supplied is just enough to balance the dissipation.

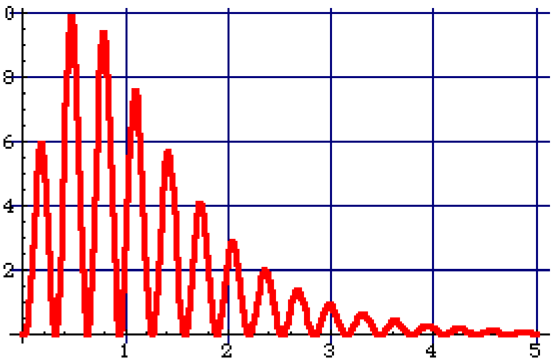

Figure 9: Difference between power supplied to the system and power dissipated by the damper.

Presumably, this excess power has been stored by the spring and the oscillating mass to sustain the oscillation. To check on this we may integrate this difference from 0 to 5 which is the range of the interval of solution for x[t].

```
NIntegrate[Evaluate[A Sin[ω t] x'[t] - b (x'[t])^2 /. s3], {t, 0, 5}, MaxRecursion → 10]
```

```
{7.97419}
```

This is about as close to 8 Joules as one can get, given that steady state has not quite been reached, as evidenced by the above figure, and from the fact that 8 Joules was a reading on figure 6.

2.2 $b = 2\sqrt{km}$ Critically Damped System

In this section we just repeat what was done in the previous one on the underdamped system. There is no need to display the computer code once more.

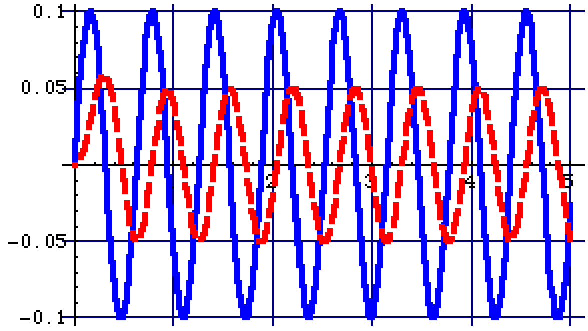

Figure 10: The dashed curve is the displacement of the mass, and the solid curve corresponds to applying the external force to the spring only.

Note that the displacement is only one half of what it would have been if the force had been applied to the spring only. The time delay between the force and the displacement is now T/4 or $\pi/2$, one quarter of a period.

2.3 Energy considerations

There is surprisingly little energy stored in the system, which is an attractive feature of a good shock absorber for example.

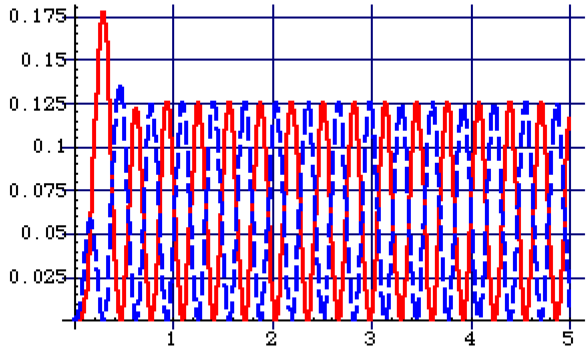

Figure 11: Potential energy: solid curve. Kinetic energy: dashed curve.

Just as for the underdamped system we calculate the difference between energy supplied by the external force and the energy dissipated in the damper.

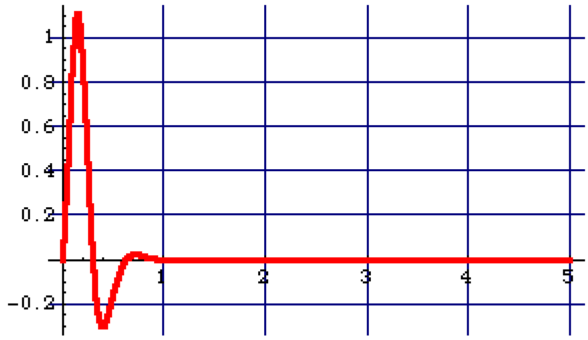

Figure 12 Difference between power supplied by the external force and power dissipated by the damper for a critically damped system.

The function in figure 12 can be integrated to find out how much energy is stored in the system, the result is in complete agreement with figure 11.

```
NIntegrate[Evaluate[A Sin[ω t] x'[t] - b (x'[t])^2 /. s4], {t, 0, 2}, MaxRecursion → 10]

{0.12503}
```

3 CONCLUSION

It has been pointed out that even in a mathematically simple oscillatory system, it is quite laborious to find particular solutions. Especially if one wants to experiment with system parameters. Once students have mastered certain mathematical techniques by hand, we believe their insight can benefit tremendously by being able to do calculations of the type that have been demonstrated. For example calculations involving eqs. (5) and (8) and their integrals will almost never be performed without access to some mathematical software. The use of such software will certainly increase. We may conclude that educators should not be concerned about these developments, rather regard it as an opportunity for students to gain further insight in the subject at hand.