Chapter 24

Contextual Learning in Math Education for Engineers

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In this paper we analyze two textbooks on differential equations according to criteria developed through educational research in mathematics education while directing attention to the needs of engineering and physical science students. The first textbook is oriented toward analytic techniques and technical competencies while the second textbook was developed to incorporate discipline-specific viewpoints from engineering and the physical sciences. Through the lens of cognitive difficulty as well as through examining the implicit messages that surface in the texts about the nature of mathematical knowledge, we conclude that a contextual approach is more in line with recommendations from the research literature.

INTRODUCTION

Educational research suggests that the manner in which a concept is presented influences which features of the concept the students find salient. For example, lecturers from engineering disciplines tend to privilege a "rate-of-change" interpretation of "derivative," while mathematics lecturers prefer a "tangent" interpretation, and students' knowledge bases reflect their instructors' preferences [1]. As with most mathematical topics, the curriculum for DEs is designed, coordinated, and taught by mathematicians. As a consequence, the presentation of key material is often couched in abstractions, general cases, and as a collection of special equations solvable by particular techniques. Engineering and physical science faculty complain that their students are not able to transfer their knowledge of DEs into their major courses. The standard curriculum is found wanting a stronger link to the physical origins of the equations being solved.

By paying attention to the equations' natural context, the theory of DEs can be bound together into a conceptually coherent whole [2]. Engineering scholars suggest incorporating a discipline-specific perspective into mathematics curricula [3, 4]. Beyond the syntax, formalism, and abstraction found in mathematics writing, the communication of mathematics is accomplished through the use of examples, nonexamples, counterexamples, exercises, problems, definitions, and theorems. The choices made by the instructor, as well as the textbook, as to which examples, and in which sequence, the students should be exposed is of utmost importance. Well-chosen examples can highlight the salient features of a mathematical idea whereas other choices would obscure the important concepts [5]. In the case of DEs, the reliance on procedural methods for solving certain equation types may obscure the links between the theory of DEs and the use of that theory.

Commercial curricula available at the K-12 level feature a wide array of instructional materials, resources, and pedagogical advice. In contrast, at the university level, the curriculum tends to be identical to the textbook. For mathematics lecturers, the textbook acts as a source of homework and exam problems; for teaching assistants, the textbook acts as a reference guide; for students, however, the textbook comprises a list of theorems, definitions, and formulae to be applied while completing assigned homework problems. Furthermore, the textbook may act as a surrogate instructor when the student is working problems or studying on his own. Since the textbook is central to many university mathematics courses, attention must be paid to the implicit messages it conveys to students and to instructors about the nature of mathematical problems [6] and the ways in which mathematical knowledge is formed [7, 8].

The present study examines and compares two textbooks on DEs, Boyce & DiPrima (8th Edition) [9] and Baker (n.d.) [10], that are currently in use at The Ohio State University (OSU) in a course for engineering and physical science majors taught in the department of mathematics. We focus only on the portions of each textbook in use at OSU. The version of [9] in use at OSU is a custom edition that covers direction fields, first-order DEs, second-order linear DEs, systems of first-order DEs, partial DEs, Fourier series, and applications. The textbook [10] was designed to address concerns voiced by engineering faculty at OSU through an emphasis on contextual learning. We selected and modified an existing conceptual framework (described below) in order to guide our analysis of these texts. In particular, the purpose of this study is twofold: (1) We examine and demonstrate each text's strengths and weaknesses according to criteria developed through educational research, and (2) Perform a research-based textbook analysis with a focus on the mathematical needs of nonmathematics majors.

This report begins by discussing perspectives on textbooks, their structure, and the assumptions of the authors who write them. In particular, we used the existing literature to determine which characteristics of textbooks are important for the teaching and learning of mathematics. We then explain the framework employed for our analysis, the methods of the analysis, and then proceed with the analysis, in which we treat the two textbooks separately. We conclude with some general notes about our findings and how they fit into the wider educational research literature.

LITERATURE REVIEW

Many textbooks adhere to a common outline: exposition-examples-exercises. The exposition presents the topic to be studied, and the examples provide paradigmatic models for the students to apply to the exercises [11]. Yet, instead of constructing a well-integrated and well-connected understanding, students tend to compartmentalize their knowledge [12]. Compartmentalization refers to inconsistent or incoherent mathematical behaviors that manifest as the apparent lack of structure in the student's knowledge base. The compartmentalization of knowledge results in a fragmented collection of rules, procedures, definitions, and theorems which are not integrated in a manner that can be brought to bear on problems outside of the setting in which they were learned. For example, students learn the formal definition of function, but may not use it when classifying relations as functions [12] or when generating examples of functions [13]. The central role of the textbook in the mathematics classroom may contribute to compartmentalization through the design and use of exercises.

In a study of common calculus textbooks, [6] classified exercises according to the level of reasoning required to find a solution. He discovered that more than 90% of the problems in each text surveyed could be solved without attending to the intrinsic mathematical properties of the task. Moreover, in 70% of the exercises, students need only search the text for a similar example and copy its solution with appropriate modifications. Thus, students may be practicing how to adapt a very small class of problems to one template rather than practicing how to apply mathematical principles to important cases.

Textbook tasks are typically phrased in predictable ways, and students pick up on these cues to help them arrive at solutions. For instance, students develop expectations about the relationships among the relevance of information provided in the problem, the relevance of context to the solution, the intended difficulty of the questions, and their past experiences [14]. These expectations need not be based on mathematical content. Indeed, students may learn to answer a question correctly without having to think about why their answer is sufficient or why the question is important [7].

These issues are due in part to the sequencing and selection of exercises, but also to the tendency of mathematics to be presented as a deductive system with formal rules of logic and syntax. Formal definitions and analytic solution techniques tend to overwhelm informal and intuitive notions [7], especially when the student is immensely successful at using a syntactic reasoning structure [15]. In order to construct a flexible and mature understanding of a mathematical topic, students must be able to coordinate their informal ideas with their formal ideas.

Compartmentalization contributes to the mismatch observed between concept image and concept definition within the student's cognitive structure [16]. A concept definition is a verbal or written definition that describes or explains the concept in a non-circular way that is accepted by the mathematical community. The student's cognitive processes, experiences, mental models, and connections associated with the concept compose the concept image and these constructions ultimately determine the meaning of the concept to the student. A robust concept image is associated with the rich connections characteristic of conceptual knowledge [17]. Tasks that encourage higher-order thinking engage students in cognitively demanding tasks such as interpreting, managing resources, constructing meaning, and using a flexible approach [18] and support concept formation.

The textbook author makes certain assumptions about his readership, about the processes involved in building a concept image, and about the instructor enacting the curriculum described by the text. These assumptions include how to communicate mathematics to the reader and in what forms mathematics should be consumed. Some authors write so as to provide a "specific, correct path through the subject matter" whereas others act "more like a general guide or critical friend" who provides resources for the reader to generate his own concepts [11, p. 388]. The author also sets the level of cognitive difficulty and determines the prominence of motivational factors such as physical applications. In developing and executing these features, the authors must bring under consideration the audience of the text and the purpose of the course it is meant to serve. These assumptions surface as implicit messages to the reader about the nature of mathematics and mathematics learning [8].

Hence, we must examine critically the features of the textbook that do and do not support student understanding. While similar analyses of textbooks have been performed in the past [7, 8], as well as fine-grained analyses of mathematical tasks [6, 19], what we report here is sympathetic to the needs of the students and faculty of engineering and physical sciences. Central to our analysis is how the approach of each text, contextual or abstract, may support or impede student cognition.

CONCEPTUAL FRAMEWORK

The role of the textbook is to communicate mathematics to the student and to provide support to the instructor. Thus, the pedagogical choices made by the textbook authors must be examined in light of what material is included, excluded, and how it is presented.

We define the cognitive demand of a task as the cognitive processes in which students must engage in order to complete a task [18, 6]. We classified tasks as requiring lower-level demands or higher-level demands using the framework developed by [19]. Lower-level tasks include memorization and performing procedures without connections. Memorization engages the student in the reproduction of previously encountered material. There is no connection to concepts or to meaning. In memorization tasks, there is no procedure applied, the solution is just a regurgitation of a fact. Tasks that require procedures without connections are characterized by a lack of ambiguity in the problem statement and an emphasis on obtaining a correct answer. These problems are typically solved through applying an algorithm without needing to understand how the procedure is connected to the underlying concepts. In these problems, the correct procedure is evident from prior experience, the placement of the task, or is specifically called for in the problem statement. These lower-level problems can be solved by recall or by a search-the-text strategy, as in [6].

Higher-level demands require a sustained level of effort, such as tasks that require the use of procedures with connections to concepts and tasks that engage the student in doing mathematics. Tasks that require procedures with connections may have solutions that coordinate multiple representations such as symbols and diagrams. These tasks are characterized by suggesting pathways to follow that are broad general procedures, but that are transparently connected to the underlying conceptual ideas. It is important to note that solutions to procedures-with-connections tasks will generally follow a procedure, but that the procedure cannot be applied mindlessly. Doing mathematics requires complex and nonalgorithmic thinking that is not explicitly called for by the task or worked out in an example. Tasks that engender mathematical practices require students to access and employ relevant knowledge and experiences, without being told which ones are relevant, and require students to analyze the task and examine constraints.

We extended and adapted the framework to analyze the expositional text and also to classify passages according to whether the passages gave high- or low-level explanations (exposition) or modeled high- or low-level tasks (examples). High-level expositions make connections to other mathematical topics, either within or external to the subject matter, or to physical interpretations. Low-level explanations are isolated and are disconnected from other subject matter. For example, a low-level explanation of a rule would refer only to another rule. A high-level explanation of that rule might also include a reference to another concept, context, informal or formal notion. Similarly, a high-level example would draw on many concepts and procedures, whereas a low-level example would display an algorithm.

METHODS

We investigated the development of a single topic, solutions to nonhomogeneous ordinary DEs, within each textbook. We built a profile of the topic in each text by examining (1) the concept definition, (2) how the concept is developed, (3) the cognitive level at which the topic is treated, and (4) the epistemological messages conveyed by the text.

We selected the texts because they are both in use at OSU, and because this report presents but one component of a study underway to compare learning outcomes of students enrolled in each model. In this report, we focus on the treatment of nonhomogeneous ordinary DEs, which are central to both the study of DEs and to its application in physical contexts. General solutions to nonhomogeneous linear equations are constructed as a sum of the solution to the associated homogeneous equation and a particular solution to the nonhomogeneous equation. The general solution will contain arbitrary constants that can be set in order to satisfy constraints such as initial, boundary, or normalizing conditions.

In Boyce & DiPrima [9], solutions to nonhomogeneous equations are discussed in sections 1.2 (Solutions of Some Differential Equations), 2.1 (Linear first-order Equations with Variable Coefficients), 2.2 (Separable first-order Equations), and 3.6 (Nonhomogeneous Second-Order Equations with Constant Coefficients; Method of Undetermined Coefficients), although they are present throughout the text, for example, in section 3.9 (Forced Vibrations). In Baker's text [10], nonhomogeneous equations are not segregated from other equations and so solutions to nonhomogeneous equations are treated heavily throughout. The most direct instruction occurs in Chapter 1 sections 1.1 (Growth and Decay), 1.2 (Forcing Effects), 1.3 (Second-Order Equations: Growth and Decay), 1.4 (Second-Order Equations: Oscillations), and 1.5 (Forcing Terms: Resonances).

We employed a text-analysis approach to generate descriptions of how each text treats nonhomogeneous equations. Each textbook is available in portable document format (.pdf). Instead of looking only at the sections where nonhomogeneous equations are discussed explicitly, we used the search feature of the .pdf viewer to help locate instances of general, homogeneous, and particular solutions within the context of solving nonhomogeneous equations. We note here that the search feature could not locate all specific instances of homogeneous and nonhomogeneous equations since textbooks will sometimes refer to equations by number. This is a delicate point since it does not allow us to accurately count the number of references to each term. To further complicate matters, the numerical references sometimes refer to the equation's content, sometimes to its form, and sometimes to its result. However, the focus of this analysis was to build a profile of the development of the target concept throughout each text with reference to student cognition, not to count the number of times the terms appear. In order to identify possibly missed references or sites of conceptual development, we carefully read each textbook section and worked the end-of-section exercises. This careful interaction with the text also helped us to gain insight into what the students might experience.

We initially classified each instance according to the role it played (definition, example, exercise, theorem, or exposition) to get a sense of how the terms were used. Many cases could not be cast into a single category since, for example, exposition could appear within a worked example. Since we were interested in the ways in which each textbook developed the concept, the instances were coded according to whether connections to other concepts were made and what connections to other concepts were possible, both in and outside of mathematics, noting any inconsistencies. This second coding allowed us to determine whether passages exhibited a high- or low-level of cognitive demand. We then examined the instances through the lenses identified in the literature search: reasoning structures, cognitive difficulty, concept definition/concept image, and epistemological grounding.

ANALYSIS AND DISCUSSION

First, we communicate the goals of each textbook in order to familiarize the reader with their purposes. Then we will address the guiding themes for each textbook in turn. Boyce & DiPrima [9] is a "traditional" text on DEs. The intention of the authors is for the text to be "widely useful" for undergraduate students of mathematics, science, or engineering by stressing "a sound and accurate (but not abstract) exposition of the elementary theory of DEs with considerable material on methods of solution, analysis, and approximation that have proved useful in a wide variety of applications" [9, p. vii]. Baker's text [10] could be classified as "reform oriented." However, unlike other reformation efforts [20], the focus is on incorporating context. [10] is intended for science and engineering majors, and draws on common problems in science and engineering fields as the motivation for creating and for solving DEs. The text uses a modeling approach and its goal is to shed light on the solution's "behavior such as the long time pattern, stationary or steady, its stability and the transition from some initial state" [10, p. ii].

Textbook [9]

The authors of [9] define homogeneous equations in multiple ways: (1) an expression in which the left hand side is a function of the independent variable, the dependent variable and derivatives of the dependent variable with respect to the independent variable and the right hand side is zero, (2) an equation of the form $dy/dx = f(x,y)$ where f can be described as a function of the ratio y/x and (3) the left hand side of a nonhomogeneous equation. Of the three definitions, (1) is used most frequently, but the three are never shown to be interchangeable. Nonhomogeneous equations are given in examples and exercises, but are not defined until the discussion on second-order linear equations with constant coefficients. Nonhomogeneous equations and linear equations, respectively, take the forms shown in equations 1 and 2:

$$L[y] = y''+p(t)y'+q(t)y = g(t) \tag{1}$$

$$f(t, y, \tfrac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y \tag{2}$$

Note that equation 1 does not have constant coefficients, despite the fact that the discussion takes place in a section on equations with constant coefficients. Of more immediate consequence is that the form of equation 2 does not reflect any of the properties of linearity that are used in the construction of general solutions to nonhomogeneous equations. These definitions are extremely limited in that they do not easily extend to more advanced notions of "nonhomogeneous" or "linear." Furthermore, when one considers that for many students, the title "linear" means "its graph has the shape of a line" it is very *unclear* why equation 2 is called linear. There is no motivation given for why linearity is a useful concept nor for why we choose to designate nonhomogeneous equations from other equations. That is, the definitions are arbitrarily chosen with little attention to the future development of the topic or to current understandings of the students.

The term "particular solution" is defined informally as referring to "some solution Y(t) of the nonhomogeneous equation" (Section 3.6, Method of Undetermined Coefficients). Prior to this usage, the text uses the phrase colloquially to refer to any specific solution of any equation. The authors do not speak of "homogeneous solutions," but rather "solutions to homogeneous equations." A general solution is one that represents all possible solutions of the DEs. It is also frequently identified as a solution that contains arbitrary constants. Sometimes the term refers to the solution of the associated homogeneous equation. In reference to the solution of a system of linear algebraic equations, the phrase "most general solution of the homogeneous system" corresponding to the nonhomogeneous system indicates that "general-ness" is a quantitative attribute. Thus, informal and formal definitions are not coordinated throughout the text.

[9] is segmented and modular and many of the chapters are written to be independent of the others. This allows for teacher flexibility in covering information in different orders, depending on the needs of the students. Sections devoted to applications are segregated from, and typically follow, sections devoted to technique. Each module covers one solution technique or one application. The examples and problems in each section represent the types of equations that can be solved with that technique. For example, separate sections expose students to equations that can be solved by the method of integrating factor and to techniques applicable to separable equations.

Theorems are stated in precise mathematical terms, in symbolic language and often resting on the theory of linear operators, as in Figure 1. The theorem in Figure 1 is proven syntactically, but many theorems in the text are proved-by-example. The theorem is used to make the proof of the form of the general solution trivial. Several important connections are missed in this theorem: (1) that only one particular solution is required to form the general solution, (2) the importance of linearity in the scope of the theorem, and (3) the impetus for subtracting the solutions.

It is interesting to note that nonhomogeneous equations are presented during the study of first-order equations. For treating those cases, [9] introduces different solution techniques (method of integrating factor, method of separation). Indeed, the authors do not mention that the method of undetermined coefficients is applicable in some of these cases as well. That is, the relationships among the concepts of homogeneity, linearity, and their attendant solution techniques are not treated in the text.

The development of the method of undetermined coefficients proceeds results-first and presents rules-of-thumb for guessing solutions. The solution procedure for linear

Theorem 3.6.1 If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (2), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{3}$$

where c_1 and c_2 are certain constants.

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t), \qquad L[Y_2](t) = g(t). \tag{4}$$

Subtracting the second of these equations from the first, we have

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \tag{5}$$

However,

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2],$$

so Eq. (5) becomes

$$L[Y_1 - Y_2](t) = 0. \tag{6}$$

Equation (6) states that $Y_1 - Y_2$ is a solution of Eq. (2). Finally, since all solutions of Eq. (2) can be expressed as linear combinations of a fundamental set of solutions by Theorem 3.2.4, it follows that the solution $Y_1 - Y_2$ can be so written. Hence Eq. (3) holds and the proof is complete.

FIGURE 1
THEOREM 3.6.1 IN [9]

nonhomogeneous equations requires that one match the form of the solution to the form of the nonhomogeneous term. This might be a point of confusion for students since there is no reason *a priori* to expect the solution to take the form of the right hand side of the equation. Viewing the equation as a process that takes an input (solution) and produces an output (the nonhomogeneous term), as is implied by the formulation in equation (1), without any justification may contribute to students' lack of understanding of the DE as a condition on the solution.

An entire section is devoted to applying the method of undetermined coefficients to models of physical systems. The only physical context used is the damped, forced mass-spring system. The homogeneous solution is described as the transient solution, and its primary purpose is stated as to "satisfy whatever initial conditions may be imposed." Physically, the energy present in the system in the form of initial position and velocity is dissipated through damping. The solution becomes the response of the system with the external force. Resonance is described as when the frequency of the forcing function is the same as the natural frequency of the system. Only periodic forcing is evident in the exercises. These descriptions provide much needed intuition about how solutions behave, but are statements of fact rather than fully integrated intuitive notions that extend to physical problems. That is, one section states facts about techniques while the next states facts about some physical systems. A connection is missing between the two that would help students realize how the two work together. Therefore, the modularity of the book contributes to low-level explanations since it divorces the conceptual interpretations from the procedures.

In each of Problems 1 through 12 find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\sin 2t$
3. $y'' - 2y' - 3y = -3te^{-t}$
4. $y'' + 2y' = 3 + 4\sin 2t$
5. $y'' + 9y = t^2e^{3t} + 6$
6. $y'' + 2y' + y = 2e^{-t}$
7. $2y'' + 3y' + y = t^2 + 3\sin t$
8. $y'' + y = 3\sin 2t + t\cos 2t$
9. $u'' + \omega_0^2 u = \cos\omega t, \quad \omega^2 \neq \omega_0^2$
10. $u'' + \omega_0^2 u = \cos\omega_0 t$
11. $y'' + y' + 4y = 2\sinh t$ *Hint:* $\sinh t = (e^t - e^{-t})/2$
12. $y'' - y' - 2y = \cosh 2t$ *Hint:* $\cosh t = (e^t + e^{-t})/2$

In each of Problems 13 through 18 find the solution of the given initial value problem.

13. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
15. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
16. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$
18. $y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

FIGURE 2
APPLY-THE-TECHNIQUE TASK IN [9]

One of the most attractive features of this textbook is the volume of tasks it offers. Assigned problem sets may vary according to the lecturer's tastes and the course goals. The tasks present at the end of any section reflect a progression from very concrete and narrow apply-the-technique tasks (Figure 2) to highly structured multi-stage tasks in which the student is asked to supply information beyond solving the DE (Figure 3) to explicitly scaffolded theory-development tasks (Figure 4). In Figure 2, the sequence of exercises increases in technical difficulty while still maintaining the same basic structure. In the second group of the exercises listed in Figure 2, the content of the equations has not changed, but the students are asked to work with initial conditions. The pattern of change from one exercise to the next is unclear and so students may not learn any cohesive concepts from such sequences. The procedure is outlined explicitly in the section summary, but in this case, the solution path cannot be memorized since the precise solution depends on the type of nonhomogeneous term. The goal of these problems is to foster procedural fluency without much attention to the conceptions students might be forming about second order linear equations. Thus these tasks are procedures-without-connections tasks.

Figure 3 shows a problem that is conceptually oriented. Students are supposed to relate the long-term behavior of the solution function to the coefficients of the dependent variable in the DE, much in the same way that solutions to a quadratic equation depend on the coefficients of the independent variable. The conclusion of problems 38 – 40 referred to in the lead-in is that "equations that are apparently very similar can have very

Behavior of Solutions as $t \to \infty$. In Problems 30 and 31 we continue the discussion started with Problems 38 through 40 of Section 3.5. Consider the differential equation

$$ay'' + by' + cy = g(t), \qquad \text{(i)}$$

where a, b, and c are positive.

30. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?
31. If $g(t) = d$, a constant, show that every solution of Eq. (i) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?

FIGURE 3
MULTI-STAGE TASK IN [9]

32. In this problem we indicate an alternate procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \quad \text{(i)}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

FIGURE 4
SCAFFOLDED THEORY PROBLEM IN [9]

different solutions." However, it is unclear in the in #30 what features or properties of the family of solutions to the given equation are captured by the condition $Y_1(t) - Y_2(t) \neq 0$. While the intention may be that the problem will build on notions of "closeness" from calculus, the exercise can be solved with purely syntactic reasoning [15]. Similarly, what is the importance of all solutions approaching *d/c* in #31? Perhaps that if the forcing term is a constant, then the long-term behavior of the solution must be constant. However, the statements of these problems are made syntactically without reference to the concepts the symbols represent. At the least, they are procedures-without-connections problems; at worst, they encourage students to rely solely on syntactic reasoning strategies that may not transfer to novel situations. In this case, the authors reduce the cognitive difficulty of the task by telling the student exactly how to approach the problem, but without giving any reason why the problem is of interest, why it should be approached that way, or what the results might mean.

The exercise in Figure 4 is meant to link the theory of DEs to the theory of linear operators through the students' knowledge of algebraic equations. In fact, viewed as a linear operator acting on the vector space of sufficiently smooth functions, the DE will partition the space into the kernel of the linear operator and the image of the linear operator. This isomorphism theorem of linear algebra is the mechanism that allows us to construct the general solution to a linear DE by summing a spanning set of homogeneous solutions and a spanning set of particular solutions. In the development of solutions to second-order linear nonhomogeneous equations, we can see a solution technique that is similar to that of solving a quadratic equation: through factoring, we reduce a second-order equation to two first-order equations. However, what is presented to the students is a set of activities reduced to a low-order. The tasks lack motivation and do not require the students to explain or justify their reasoning, as they are asked only to verify information.

In each of the problem sets discussed, the procedure is explicitly called for or is evident from the statement or structure of the task. Even in the multi-stage and theory-driven problems, the step-by-step instructions eliminate the sense-making aspects of the task [18]. Furthermore, there is little to these problems that is unique to the study of DEs. The path to the solution is consumed by tedious algebra or the application of first- and second-semester calculus techniques.

Overall, the authors employ a top-down approach to *giving* students mathematics. The dominant metaphor is that mathematics is "covered," evidenced by the authors' intention to present a broad range of topics and expect a wide range of competencies. The text communicates the strong epistemological commitment that knowledge is derived through synthetic reasoning structures. Problems that are straightforward to execute are valued and the doing of mathematics is reinforced as practicing solution techniques to technical problems. This can result in students thinking that mathematics is complete and that all the solutions are known. Indeed, they may "form the impression that there is an enormous amount to know, and that experts already know it all, when what society wants (or claims to want) is that each individual learn to enquire, to weigh up, to analyse, to conjecture, and to draw and justify conclusions" [21, p. 107].

Textbook [10]

Ref. [10] is organized by context and follows an example-exposition-exercise format. In each section, a detailed discussion of one or two application-oriented examples is followed by an abstract mathematical view of the material. The text is integrated, rather than modular, with contextual examples motivating conceptual development. While this structure improves the continuity of the text's narrative, it may prove more difficult for students to use it as a quick reference. Instead of theorems, the text gives general "principles." Similar to [9], some of the principles are proved and some are justified through examples. There are 19 principles through the text, cataloged in an appendix. Each of the principles is stated in a mixed symbolic/verbal language so that the content of each statement is made transparent while maintaining the abstract nature of a generalizable theorem. For example, Principle 7, shown in Figure 5, underlies the method of undetermined coefficients and it is this principle that fuels all solution techniques taught. This approach to mathematical formalization may help students appreciate the need for abstract mathematical statements by not divorcing the semantic meaning with the symbolic statements.

In [10], a solution to a DE is any function that can be substituted into the equation and the equation still makes sense (Principle 2). Solutions are educated guesses at functions that might make sense when substituted into the equation. [10] does not focus the students on memorizing techniques or conditionals, but encourages them to make and justify guesses. Every solution is a conjecture until it is proven to make sense. A nonhomogeneous equation is defined as describing a system in the presence of forcing. Particular and homogeneous solutions are defined concomitantly, as seen in Figure 6 (p. 15).

The definition is motivated by the physical response of a system to an external force and so each part of the solution is characterized by its dual role in satisfying the DE and its referent in the physical system. The particular solution "takes care of" the forcing, but it cannot satisfy the initial conditions. Instead of defining the general solution as one with arbitrary constants, [10] presents it as a combination of functions that each to their part to satisfy constraints imposed by the system.

> Exponential solutions of the form $\exp(\lambda x)$ to homogeneous, linear differential equations with constant coefficients will lead to a polynomial in λ, called the characteristic polynomial. The degree of the polynomial will match the order of the differential equation. The roots of the polynomial will determine the solution.

FIGURE 5
PRINCIPLE 7 IN [10]

> From a mathematical perspective, different forcing terms will produce different responses, and if the system is described by a differential equation, then this means we should expect different solutions to the differential equation. These solutions will be called particular or inhomogeneous solutions. There is still the evolution of the system from some initial state, which would occur even in the absence of forcing. This free response of the system is reflected in the presence of homogeneous solutions and we saw several examples in the previous section.

FIGURE 6
CONCEPT DEFINITION IN [10]

Another prominent feature of the passage in Figure 6 is that of setting expectations. Instead of stating why objects must or must not have certain characteristics, the author communicates expectations. Phrases such as "we expect" or "we anticipate," followed by checking outcomes, demonstrate conjecture and sense-making. As a pedagogical tool, the author manages expectations to relate informal reasoning structures into formal ideas. Thus, the dominant metaphor of the text is "sense-making."

Ref. [10] maintains a connection between the system's response to forcing and the particular solution of a nonhomogeneous equation -- the response mimics the force -- throughout the text. This connection is used to explain growth and decay, resonance, and the effects of boundary conditions in partial DEs. The text makes explicit that general solutions are constructed through linear combinations of a homogeneous solution (an element of the linear operator's null space, a response to initial conditions) and a particular solution (a representative of the equivalence class of functions mapped to the nonhomogeneous term, the system response to forcing). In the case of a two-point boundary value problem, such as $u''+u = f(t)$ with $u(0) = 0$, $u(L) = 0$, the eigenfunctions are described as independent homogeneous solutions of the associated homogeneous equation. A linear combination of all the eigenfunctions must be taken in order to get a general solution, which is the Fourier series of f. By immediately, and consistently, representing the relationship among particular, homogeneous, and general solutions, a foundation is laid that extends to advanced topics.

The method of undetermined coefficients is introduced in Section 1.2 (Forcing Effects) for first-order linear equations in the context of a mixing problem. The procedure is developed through making guesses and balancing coefficients. There are no general formulas for solutions in [10]. Thus, it is unlikely that students can complete the homework problems in each section using only the ultimate formula or procedure. However, it does not examine which guesses may be appropriate for nonlinear equations. The text does treat systems of nonlinear ODEs through a process of approximation and linearization. This approach, and the experiences it provides, may be useful for students in the physical sciences, but it does limit the types of equations that can be solved without more advanced techniques.

There are far fewer problems in [10], and they tend to be less technically difficult. Almost all problems are "word problems," but some task sequences do focus on skill acquisition. It is not unusual for tasks in [10] to admit solutions that are several pages in length. In comparison, the [9] exercises require far more attention to the details of integrating complicated functions. Typical "word problems" appear in multiple contexts ranging from chemical reactions, to objects falling with drag, to temperature gradients, and require students engage in complex, nonalgorithmic thinking. For example, consider the problem in Figure 7.

> A cooling fin is designed to allow heat to conduct along a thin plate while heat is transport [sic] to the ambient air through Newton's law of cooling that states the flux of heat is proportional to the difference in temperature between the fin and the air...Instead of two initial conditions, we have one condition at $x = 0$ and one at $x = L$ (these conditions are called boundary conditions). Use these conditions to find the solution for the temperature profile. In particular, find the temperature at the end $x = L$. The parameter K depends on the material of the plate, whereas H depends on the exposed area of the plate. Estimate a value for HL^2/K that will make the temperature at the end of the plate close to T_0.

FIGURE 7
A HIGH-LEVEL PROBLEM IN [10]

This problem is messy. It requires students to deal with ambiguity, make assumptions, and tie the outcomes back to the context of the problem. The problem in Figure 7 requires the solution of a two-point boundary value problem, and the task is fully integrated into the mathematical context of second-order linear equations with constant coefficients. Thus, later discussions about boundary value problems or heat conduction in a rod may be built on the informal knowledge about temperature profiles built here. The key feature of these sequences of tasks is that they are related to the examples worked in the text, but are not identical. This fact, coupled with the varying contexts, maintains a high level of cognitive demand.

Through detailed examples, the author makes assumptions as the solution develops, instead of in the problem statement. This serves two pedagogical purposes: detailing why certain mathematical formalisms are necessary and modeling problem solving behavior. Taken together, these features indicate that the author conceptualizes mathematics as sense-making and values intuition over technical competencies. Thus, [10] shares forms of knowledge that enable students to approach complex problems and to make decisions in the course of solving them.

The concepts associated with solving a nonhomogeneous DE are built upon ideas with which the students are already familiar. Thus their informal and common-sense knowledge structures can be coordinated with the formal, and more versatile, mathematical language [22]. Embedding problems in contexts about which students have real, experiential intuition may help support mathematical language development and associate already-constructed, and newly constructed, mental images and knowledge structures with symbols. That is, the structure of the text may help students generate meaning for the syntax.

Throughout [10], the problems increase in complexity, not just in technical difficulty, and so cognitive resources are consumed by higher-order tasks instead of by computation. In solving such problem sequences, students must recognize not only how, but also when, to use procedures and concepts when approaching a problem [23]. [10] communicates the message that mathematical knowledge is derived through understanding the relationships between the mathematical system and its referent. Since mathematical development is coordinated with physical intuition, students are encouraged to develop conceptual knowledge that is more richly connected and robust. Experiencing mathematics in context and building a more robust concept image may serve to facilitate transfer to other contexts.

CONCLUSIONS

The purpose of this study was to analyze two textbooks' treatments a topic fundamental to the study of DEs according to criteria that are drawn from educational research and from theoretical perspectives. In particular, our intention was to highlight features that are and are not compatible with research in mathematics education. We did not, however, examine linguistic elements, reader cues, nor other features of textbooks that help students locate meaning within a written text.

We found that [9] encourages a wide range of syntactical competencies, and introduces students to a wealth of techniques they can use to construct analytic solutions to DEs. On the other hand, [10] was written in response to criticism of a course on DEs voiced by engineering faculty [24] and so the content of [10] is organized around the concepts and principles of applied mathematics

The inconsistent and inaccurate concept definitions found in [9] form unsturdy foundational knowledge. The modular organization of the text, the authors' choices for how to communicate examples, expositions, and exercises reinforce the authors' epistemological commitment that mathematics knowledge is a collection of disconnected, symbolic competencies. Ideas are not connected and the students do not have to decide how to approach a problem. Taken together, these indicators provide evidence that the text is written at a low-level of cognitive difficulty. This is not to say that the material or the tasks are not difficult, but previous research suggests that low-level tasks may not support the construction of the types of mathematical knowledge desired in settings outside the mathematics classroom [25, 26]. The systematic proceduralizing encouraged by [9] may in fact contribute to the compartmentalization of knowledge [27].

The definitions for homogeneous, nonhomogeneous, general, and particular solutions are stated at the beginning of [10] and are used consistently throughout the text. The concept is developed through many examples and exercises of varying complexity. Since the concept is used to build the application of DE theory, students are encouraged to coordinate their formal and informal ideas about representing systems mathematically. The topic is consistently treated at a high cognitive level, while still attending to the development of procedural fluency in deriving analytic solutions. As evidenced by the text, the author views mathematics as a means to describe and formalize patterns within a system.

One should note that low-level cognitive tasks are not undesirable. Indeed, automatizing procedures is desirable behavior [28], since it requires that the student notice and use patterns [29]. Moreover, low-level tasks often provide entry into a new topic. However, what we have found from this fine-grained theoretical analysis is that an applications-oriented approach to DEs renders visible the connections among mathematical concepts, mathematical procedures, and physical applications, and aligns more clearly with desirable traits discovered through educational research.

Of course, an applications-oriented or research-based approach is not in itself sufficient for causing learning in students. One must also consider the needs of individual students and the methods of classroom instruction and assessment. Research has shown that both increased cognitive difficulty and rich tasks are linked with increased learning [30], higher-order thinking skills [31, 32], persistence [33], and enjoyment of mathematics [33, 34]. Given the centrality of the textbook to undergraduate mathematics classrooms, one cannot deny the need to examine carefully what messages they might carry for students.

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